Triangle Inequality for Altitudes

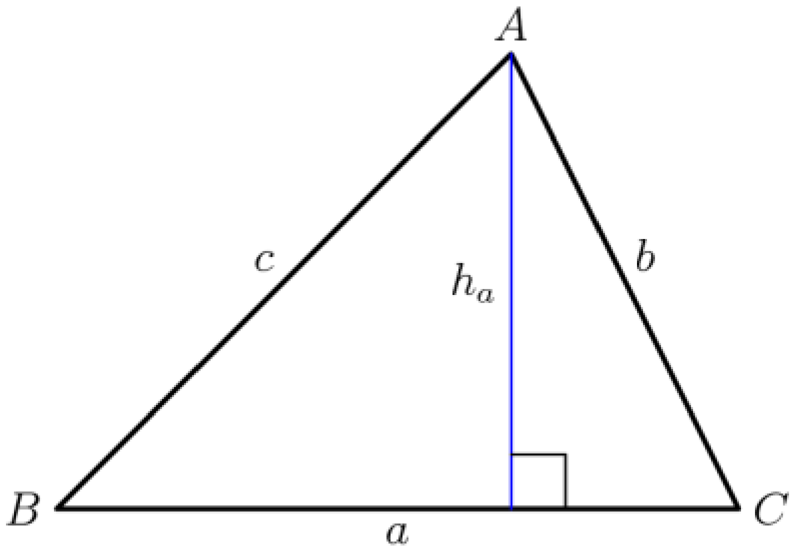
In basic geometry, almost everyone learns the *Triangle Inequality*, which states that the sum of any two sides in a triangle must be greater than the third side. Algebraically, if the sides of a triangle are a, b, and c, then we have the set of inequalities

$$\begin{aligned} a+b &> c \\ b+c &> a \\ c+a &> b. \end{aligned}$$

The triangle inequality shows up all over the place. There are vector versions of it once you learn a little more math, and it gets generalized even more once you learn about metric spaces in topology.

But, I won't go too deep into that today. This is just a short post describing how the typical triangle inequality used in plane geometry also works for altitudes in a triangle.

An altitude of a triangle, (or a "height" in more lax language) is a segment that goes from one vertex in a triangle to the opposite side, and is perpendicular to that opposite side.



The whole reason we care about an altitude is because it helps us find the area. The area of a triangle is half the product of a base and the height. The length of the altitude is the height, and the segment it is perpendicular to is the base. So, the area of the triangle above is $\frac{1}{2}a \cdot h_a$.

But this is true for any of the bases and their altitudes. So, of the altitude from vertex B to side b is h_b, the area is also $\frac{1}{2}b \cdot h_b$, and similarly we would have $\frac{1}{2}c \cdot h_c$. Let's say the area is some number R. Then, we can rearrange each of the area expressions to say

$$\begin{aligned} a &= \frac{2R}{h_a} \\ b &= \frac{2R}{h_b} \\ c &= \frac{2R}{h_c} \end{aligned}$$

Now, just substituting the expressions on the right into our original set of triangle inequalities (and removing the common factor of $2R$ from each one), we get

$$\begin{aligned}
\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c} \\
\frac{1}{h_b} + \frac{1}{h_c} > \frac{1}{h_a} \\
\frac{1}{h_c} + \frac{1}{h_a} > \frac{1}{h_b}
\end{aligned}$$

And so the triangle inequality for altitudes is a *harmonic* version of the triangle inequality for the sides of a triangle. This is reminiscent of resistors in series versus those in parallel; in the first case, you just sum the resistances. In the second case, you sum their reciprocals to get the reciprocal of the total. It's fun to find these little symmetries in math.